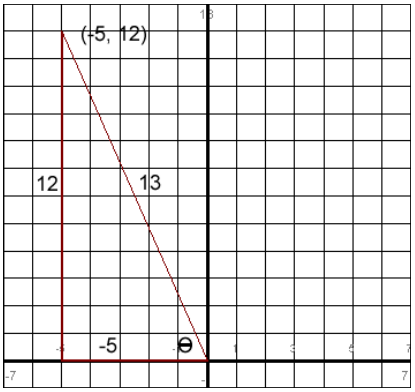
5.4 Trigonometric Functions of General Angles

EXAMPLE: A point on the terminal side of angle θ is (–5, 12). Find the exact value of each of the six trigonometric functions of θ.

First we want to plot the point and then form a triangle like below:



To get the hypotenuse, we need to use the Pythagorean theorem, which is $a^2 + b^2 = c^2$. Then solve for c: $(12)^2 + (-5)^2 = c^2$. The we have: $144 + 25 = c^2$. Solving for c you will get 13. The angle θ is labeled on the graph. It is at the origin. We know that the hypotenuse is 13. The opposite side is 12, and the adjacent side is –5. From this we can get our six trigonometric values.

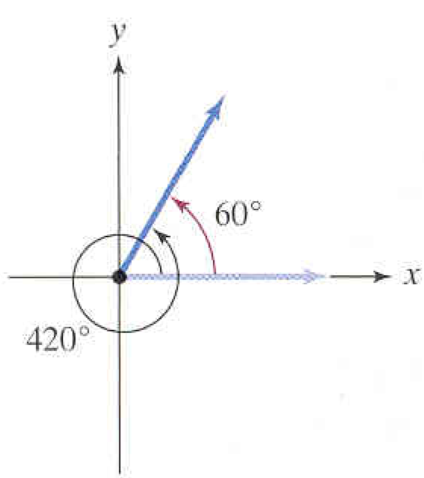
$\sin\theta = \frac{12}{13}$ $\qquad$ $\csc\theta = \frac{13}{12}$

$\cos\theta = \frac{-5}{13}$ $\qquad$ $\sec\theta = \frac{13}{-5}$

$\tan\theta = \frac{12}{-5}$ $\qquad$ $\cot\theta = \frac{-5}{12}$

EXAMPLE: Draw 420° in standard position. Then find $\sin 420^\circ$.

From a previous section we know that one revolution is equal to 360 degrees. If we subtract: 420 – 360 we get 60 degrees, which means this goes around one full revolution and then we need to go an extra 60 degrees:



Now let's find $\sin 420^\circ$. We don't have 420° on our table, but we can write $\sin 420^\circ$ as $\sin(60^\circ + 360^\circ)$ From our picture we see that this is the same as $\sin 60^\circ$. From our table $\sin 60^\circ = \frac{\sqrt{3}}{2}$, so we can conclude $\sin 420^\circ = \frac{\sqrt{3}}{2}$.

This conclusion tells us that $\sin 420^\circ = \sin(60^\circ + 360^\circ) = \sin 60^\circ$. This means that it doesn't matter how many revolutions you make because you will end up at the same point. Formally the rules say:

$\sin(\theta + 360k) = \sin\theta$ $\qquad$ $\csc(\theta + 360k) = \csc\theta$

$\cos(\theta + 360k) = \cos\theta$ $\qquad$ $\sec(\theta + 360k) = \sec\theta$

$\tan(\theta + 360k) = \tan\theta$ $\qquad$ $\cot(\theta + 360k) = \cot\theta$

EXAMPLE: Find the exact value without using a calculator: $\cos 405^\circ$.

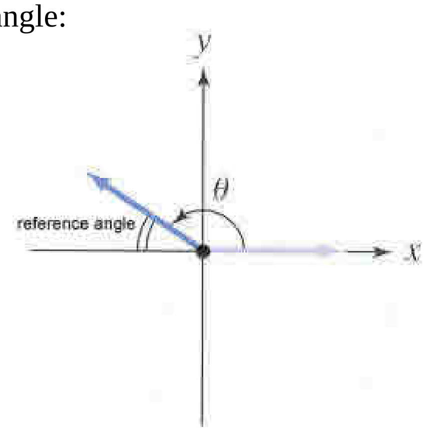
We can rewrite this as $\cos(45^\circ + 360^\circ)$. From our rules above this says that $\cos(45^\circ + 360^\circ) = \cos 45^\circ$. From our table this is equal to $\frac{\sqrt{2}}{2}$. Therefore, $\cos 405^\circ = \frac{\sqrt{2}}{2}$.

EXAMPLE: Find the exact value without using a calculator: $\tan 780^\circ$.

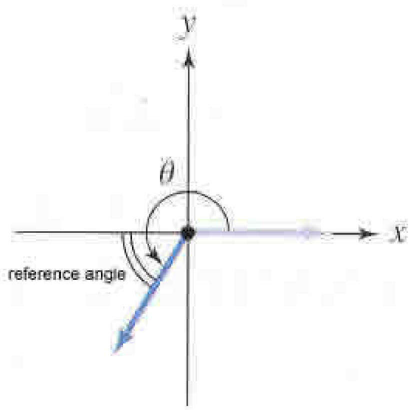
If we divide 780 by 360 we get 2 with a remainder of 60. This means we can rewrite 780 as 60 + 360(2). So now we can write $\tan(60^\circ + 360^\circ(2))$. From our rules above this says that this is equal to $\tan 60^\circ$ From our table this is equal to $\sqrt{3}$. Therefore, $\tan 780^\circ = \sqrt{3}$.

Notice so far all our resulting angle were something between 0 and 90. What if we end up with something between 90 and 360 degrees? This is where you want to use:

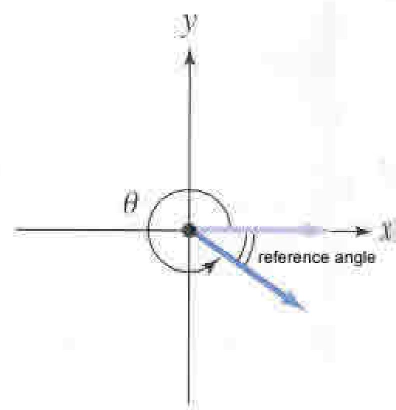
Reference Angle – an angle between 0 and 90 that is formed by the terminal side of an angle and the x-axis. The reference angle is labeled below. It is indicated by the double curved lines. Notice that no matter where the angle is drawn it is measured from the x-axis. Under each drawing it tells you how to find the reference angle:



Ref. angle = $180^\circ - \theta$



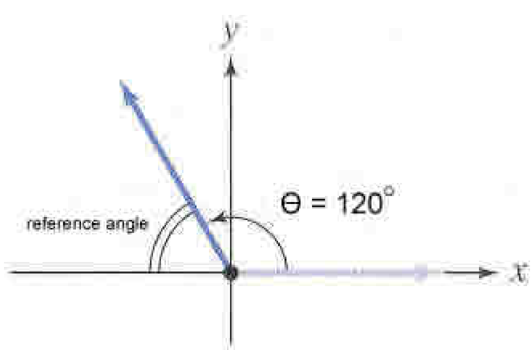
Ref. angle = $\theta - 180^\circ$



Ref. angle = $360^\circ - \theta$

EXAMPLE: Draw 120° in standard position and then find its reference angle.

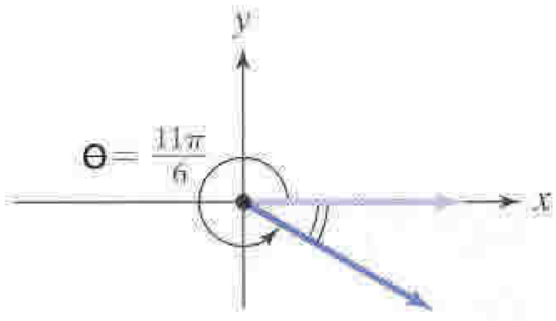
First we will draw it in standard position. The reference angle is indicated by the double curved lines:



To find the reference angle, we use the formula above, which says that the reference angle is $180^\circ - \theta$. So we our reference angle is: $180^\circ - 120^\circ = 60^\circ$.

EXAMPLE: Draw $\frac{11\pi}{6}$ in standard position and then find its reference angle.

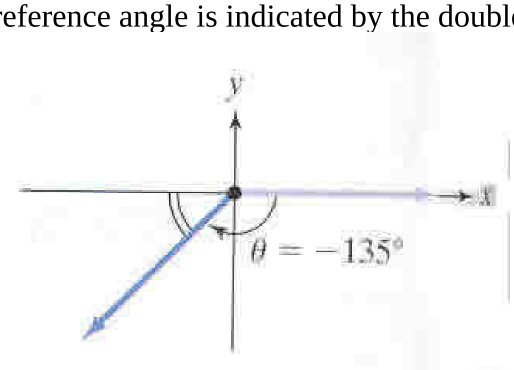
We can change $\frac{11\pi}{6}$ into degrees so we know how to graph it: $\frac{11\pi}{6}\cdot\frac{180}{\pi}=330^{\circ}$. Now we will draw it in standard position. The reference angle is indicated by the double curved lines



To find the reference angle, we use the formula above, which says that the reference angle is $360^{\circ}-\theta$. So we our reference angle is: $360^{\circ}-330^{\circ}=30^{\circ}$. We need to change this back into radians since the problem was originally given in radians.   Our reference angle is: $\frac{\pi}{6}$.

EXAMPLE: Draw -135° in standard position and then find its reference angle.

Remember when we draw it in standard position we must go clockwise this time since the angle is negative. The reference angle is indicated by the double curved lines:



To find the reference angle, we use the formula above, which says that the reference angle is $180^{\circ}-\theta$. So we our reference angle is: $180^{\circ}-135^{\circ}=45^{\circ}$.

Sign values of sine, cosine, and tangent in each quadrant

$\sin\theta$ + $\cos\theta$ − $\tan\theta$ −	$\sin\theta$ + $\cos\theta$ + $\tan\theta$ +
$\sin\theta$ − $\cos\theta$ − $\tan\theta$ +	$\sin\theta$ − $\cos\theta$ + $\tan\theta$ −

Depending on which quadrant you are in the sine, cosine, and tangent functions will be either positive or negative. You will need this for using reference angles to find trigonometric values.

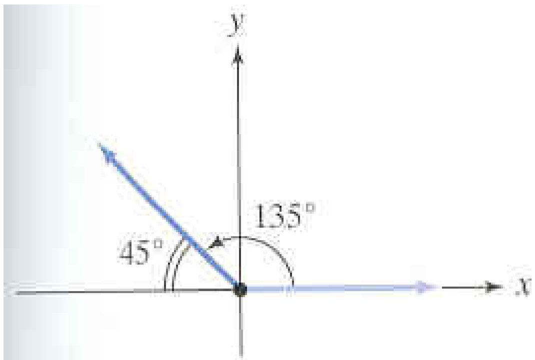
The quadrants are number from 1 to 4 counterclockwise starting with the upper right quadrant. Each quadrant has a certain angle value: In quadrant 1: $0<\theta<90^{\circ}$, in quadrant 2: $90<\theta<180$, in quadrant 3: $180<\theta<270$, and in quadrant 4: $270<\theta<360$.

How to find the trigonometric value for any angle:

1.) Find the reference angle.
2.) Apply the trig function to the reference angle
3.) Apply the appropriate sign.

EXAMPLE: Find the exact value of $\cos 135^\circ$ using reference angles. Draw the angle in standard position and indicate the reference angle.

We will follow the three steps from above.



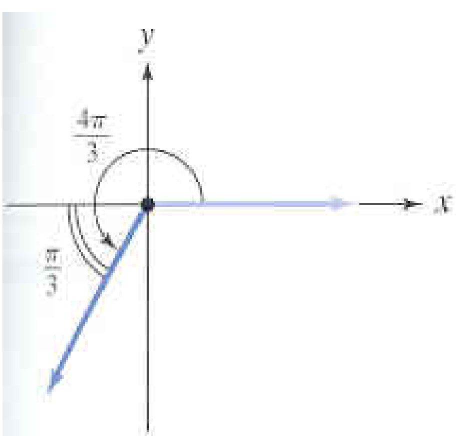
1.) First we will draw this angle in standard position. The reference angle is indicated by the double curved lines.
We found the reference angle by taking $180^\circ - 135^\circ = 45^\circ$

2.) We need to apply the trig function to our reference angle, so we will do $\cos 45^\circ = \frac{\sqrt{2}}{2}$.

3.) We need to apply the appropriate sign. This is where we will use the sign chart from the last page. This angle is in the second quadrant, so cosine needs to be negative here. So now we can write our answer: $\cos 135^\circ = -\frac{\sqrt{2}}{2}$.

EXAMPLE: Find the exact value of $\sin \frac{4\pi}{3}$ using reference angles. Draw the angle in standard position and indicate the reference angle.

We can change this into degrees to see what quadrant we are in: $\frac{4\pi}{3} \cdot \frac{180}{\pi} = 240^\circ$

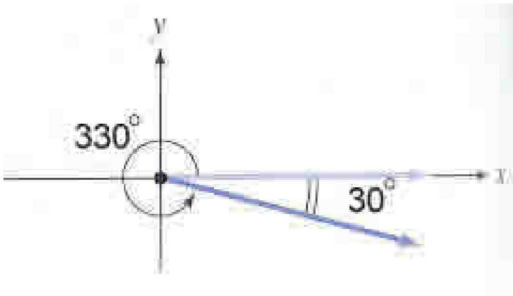


1.) First we will draw this angle in standard position. The reference angle is indicated by the double curved lines.
We found the reference angle by taking $240^\circ - 180^\circ = 60^\circ$. This is equivalent to $\frac{\pi}{3}$.

2.) We need to apply the trig function to our reference angle, so we will do $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

3.) We need to apply the appropriate sign. This angle is in the third quadrant, so sine needs to be negative here. So now we can write our answer: $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$.

EXAMPLE: Find the exact value of $\cos 330^\circ$ using reference angles. Draw the angle in standard position and indicate the reference angle.



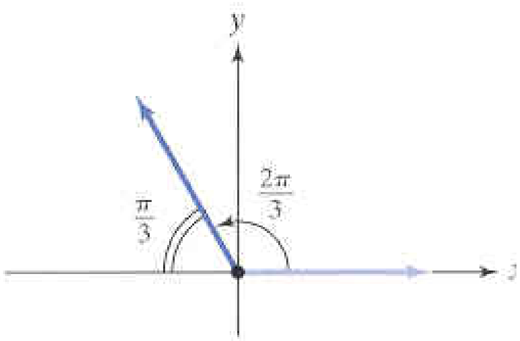
1.) First we will draw this angle in standard position. The reference angle is indicated by the double curved lines.
We found the reference angle by taking $360^\circ - 330^\circ = 30^\circ$.

2.) We need to apply the trig function to our reference angle, so we will do $\cos 30^\circ = \frac{\sqrt{3}}{2}$.

3.) We need to apply the appropriate sign. This angle is in the fourth quadrant, so cosine needs to be positive here. So now we can write our answer: $\cos 330^\circ = \frac{\sqrt{3}}{2}$.

EXAMPLE: Find the exact value of $\tan \frac{14\pi}{3}$ using reference angles. Draw the angle in standard position and indicate the reference angle.

We can change this into degrees: $\frac{14\pi}{3} \cdot \frac{180}{\pi} = 840^\circ$. We can subtract two revolutions from this:
$840^\circ - 360^\circ - 360^\circ = 120^\circ$. From our rules, we can rewrite this problem as: $\tan(120^\circ + 360(2)) = \tan 120^\circ$
In radian form it would look like this: $\tan \frac{2\pi}{3}$.



1.) First we will draw this angle in standard position. The reference angle is indicated by the double curved lines.
We found the reference angle by taking $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$. (180 – 120)

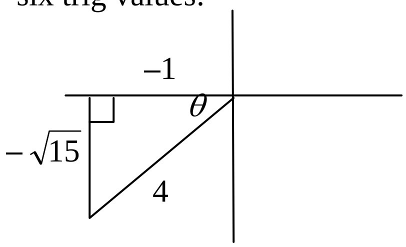
2.) We need to apply the trig function to our reference angle. We have $\tan \frac{\pi}{3} = \sqrt{3}$.

3.) We need to apply the appropriate sign. This angle is in the second quadrant, so tangent needs to be negative here. So now we can write our answer: $\tan \frac{14\pi}{3} = -\sqrt{3}$.

EXAMPLE: Given $\cos\theta = -\frac{1}{4}$ and $180^\circ < \theta < 270^\circ$, find the exact value of the six trig functions.

First we need to draw the triangle like we did the previous section. This time we are told $180 < \theta < 270$, which means we need to draw the triangle in the third quadrant. Our fraction is negative. That means that either 1 or 4 must be negative when we put this in our drawing. The hypotenuse is NEVER negative, so this means that 1 must be negative since this is the adjacent side. Our θ is drawn at the origin, and this is always where it will be drawn. This is like a reference angle.

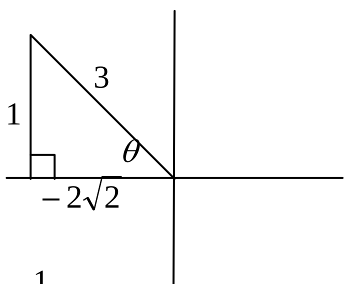
We can use the Pythagorean theorem to find the missing side: $a^2 + (-1)^2 = 4^2$. Solving this you will get $a = \pm\sqrt{15}$. In our drawing, since we are in the third quadrant, we MUST use the negative answer. The reason why is this vertical distance is really a y value, and if we think about it in terms of graphing something, the y would be negative since we are below the x-axis. So now our drawing is complete and we can find the six trig values:



$\sin\theta = -\frac{\sqrt{15}}{4}$, $\csc\theta = -\frac{4}{\sqrt{15}} = -\frac{4\sqrt{15}}{15}$, $\cos\theta = -\frac{1}{4}$, $\sec\theta = -4$, $\tan\theta = \sqrt{15}$, $\cot\theta = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15}$

EXAMPLE: Given $\csc\theta = 3$ and $\frac{\pi}{2} < \theta < \pi$, find the exact value of the six trig functions.

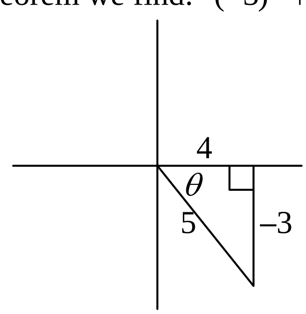
We are in the second quadrant. We can rewrite the original problem as $\csc\theta = \frac{3}{1}$. Now we know that the hypotenuse is 3 and the opposite side is 1. Using the Pythagorean theorem we can find the third side: $a^2 + 1^2 = 3^2$. Solving this we get $a = \pm\sqrt{8}$ which can be written as $a = \pm 2\sqrt{2}$. Since we are in the second quadrant we want to choose $-2\sqrt{2}$ since in the second quadrant the x value is negative.



$\sin\theta = \frac{1}{3}$, $\csc\theta = 3$, $\cos\theta = -\frac{2\sqrt{2}}{3}$, $\sec\theta = -\frac{3}{2\sqrt{2}} = -\frac{3\sqrt{2}}{4}$, $\tan\theta = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$, $\cot\theta = -2\sqrt{2}$

EXAMPLE: Given $\tan\theta = -\frac{3}{4}$ and $270^\circ < \theta < 360^\circ$, find the exact value of the six trig functions.

This will be drawn in the fourth quadrant, so the opposite side must be negative. By the Pythagorean Theorem we find: $(-3)^2 + (4)^2 = c^2$. So c = 5. Remember the hypotenuse is ALWAYS positive.



$\sin\theta = -\frac{3}{5}$, $\csc\theta = -\frac{5}{3}$, $\cos\theta = \frac{4}{5}$, $\sec\theta = \frac{5}{4}$, $\tan\theta = -\frac{3}{4}$, $\cot\theta = -\frac{4}{3}$